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# Ovals and unitals in commutative twisted field planes

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**Abstract**

It is known that commutative twisted field planes of odd order have orthogonal polarities, and, for square orders, also unitary polarities, see Dembowski (Finite Geometries, Springer, Berlin, 1968). In this paper, the associated ovals and unitals are considered and their collineation groups are determined. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. A model for a commutative twisted field plane

Let GF($q$) be a Galois field of order $q = p^m$, $p$ odd prime, containing a subfield GF($d$) such that $(-1)$ is not a $d-1$ power in GF($q$). Let $d = p^s$, and put $r = m/s$. Then $r \geqslant 3$ is odd, and every element $x \in$ GF($q$) can be uniquely expressed as $x = a^d + a$, with $a \in$ GF($q$).

The commutative twisted field $Q(+,*)$ of order $q$ may be obtained from GF($q$) by replacing the multiplication in GF($q$) with a new one defined by the rule: $(a^d + a) * (b^d + b) = a^d b + a b^d$, see [2, 5.3.7].

Correspondingly, the affine plane $\pi$ coordinatized by $Q(+,*)$ may be obtained from the desarguesian plane AG$(2,q)$ over GF($q$) by replacing lines not through $(\infty)$ with graphs of functions over GF($q$). These functions are $y = m^d x + m x^d + w$ where $m, w$ range over GF($q$).

Translations of AG$(2,q)$ are also translations of $\pi$. Moreover, for every $a, b, c \in$ GF($q$), we have a collineation $x' = x + a$, $y' = c^d x + c x^d + y + b$ which is either a

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translation or the product of a translation with a shear with special point $(\infty)$, according as $c=0$ or $c\neq 0$. The group $\Pi$ consisting of these collineations is a metabelian normal subgroup of order $q^3$ of the full collineation group $\Gamma$ of $\pi$. More precisely, $\Pi$ admits a complement $\mathscr{A}$ in $\Gamma$ which actually coincides with the stabilizer of the points $(0,0)$, $(0)$ and $(\infty)$ of the fundamental triangle of $\pi$. Moreover, $\mathscr{A}$ is isomorphic to the autotopism group of the multiplicative loop of $Q(+,*)$. For this reason, $\mathscr{A}$ is also called the autotopism collineation group of $\pi$. In [1] Albert showed the solvability of $\mathscr{A}$ and also gave a method of listing all elements of $\mathscr{A}$.

In Sections 2 and 3, we need some more results on $\mathscr{A}$, especially an explicit description of its structure, as given in Proposition 2 depending on some particular collineations which we introduce in

**Proposition 1.** (I) *The following mappings*

$$x'=ux, \qquad y'=u^{d+1}y, \quad u\in \mathrm{GF}(q)^*, \tag{1}$$

$$x'=x, \qquad y'=by, \quad b\in \mathrm{GF}(d)^*, \tag{2}$$

$$x'=cx, \qquad y'=y, \quad c\in \mathrm{GF}(d)^*, \tag{3}$$

$$x'=x^{\sigma}, \qquad y'=y^{\sigma}, \quad \sigma\in \mathrm{Aut}\,\mathrm{GF}(q) \tag{4}$$

*are* (*autotopism*) *collineations of* $\pi$.

(II) *The* $((0),[0])$*-homologies of* $\pi$ *are mappings with Eq.* (3).

**Proof.** We only prove (II). All the rest is a straightforward computation. Let the mapping $x'=a*x$, $y'=y$ be a $((0),[0])$-homology of $\pi$. Putting $x'=X'^d+X'$, and $a=A^d+A$, the equation $x'=a*x$ becomes

$$X'^d+X'=A^dX+AX^d \tag{5}$$

and we must prove that $A^d=A$.

From Eq. (5),

$$X'^{d^k}+X'^{d^{k-1}}=A^{d^k}X^{d^{k-1}}+A^{d^{k-1}}X^{d^k} \quad \text{for each } 1\leqslant k\leqslant r.$$

Hence,

$$\sum_{k=1}^{r}(-1)^{k+1}(X'^{d^k}+X'^{d^{k-1}})=\sum_{k=1}^{r}(-1)^{k+1}(A^{d^k}X^{d^{k-1}}+A^{d^{k-1}}X^{d^k}).$$

It is easily seen that this implies

$$\begin{aligned}2X'=&(A^d+A^{d^{r-1}})X+(A-A^{d^2})X^d+(A^{d^3}-A^d)X^{d^2}+\cdots\\&+(A^{d^{r-3}}-A^{d^{r-1}})X^{d^{r-2}}+(A^{d^r}-A^{d^{r-2}})X^{d^{r-1}},\end{aligned} \tag{6}$$

$$\begin{aligned}2X'^d=&(A^d-A^{d^{r-1}})X+(A^{d^2}+A)X^d+(A^d-A^{d^3})X^{d^2}+(A^{d^4}-A^{d^2})X^{d^3}+\cdots\\&+(A^{d^{r-2}}-A^{d^r})X^{d^{r-1}}.\end{aligned} \tag{7}$$

Assume that the mapping $x' = a * x,\ y' = y$ defines a collineation of $\pi$. Since this mapping fixes the point $(0,0)$, it leaves the pencil of lines through $(0,0)$ invariant.

Thus, for each $M \in \mathrm{GF}(q)$, there exists $M' \in \mathrm{GF}(q)$ such that

$$M'^d X' + M' X'^d - (M^d X + M X^d) = 0 \tag{8}$$

for every $X \in \mathrm{GF}(q)$. By virtue of Eqs. (6) and (7), (8) is equivalent to

$$\begin{aligned}
&[M'^d(A^d + A^{d^{r-1}}) + M'(A^d - A^{d^{r-1}}) - 2M^d]X \\
&\quad + [M'^d(A - A^{d^2}) + M'(A^{d^2} + A) - 2M]X^d \\
&\quad + [(M'^d - M')(A^{d^3} - A^d)]X^{d^2} + \cdots + [(M'^d - M')(A^{d^r} - A^{d^{r-2}})]X^{d^{r-1}} = 0.
\end{aligned}$$

As $d^{r-1} < q$, each coefficient must vanish. Since $r \geqslant 3$, $A = A^{d^2}$ (and $AM' = M$) holds. But $\mathrm{GF}(q)$ has no subfield of order $d^2$, so we infer that $A = A^d$. $\square$

**Remark 1.** From the final part in the previous proof we can also infer that $a \in \mathrm{GF}(d)^*$. It is worth mentioning that this together with a known theorem on the nuclei of a semifield (see [5, Theorem 8.2(c)]) gives an apparently new result on the middle nucleus of the commutative twisted field: the middle nucleus $\mathcal{N}_m$ consists of the elements of the subfield $\mathrm{GF}(d)$.

**Remark 2.** Since the full collineation group of a semifield plane is soluble if and only if the autotopism group $\mathcal{A}$ is, the solvability of $\mathcal{A}$ together with the classification of solvable 2-transitive permutation groups (see [6, XII Theorem 7.3]) has the following immediate corollary: If $G \leqslant \mathcal{A}$ and $I$ is a $G$-invariant point-set of size $p^h$ such that $G$ acts on $I$ as a 2-transitive permutation group $\bar{G}$ containing a subgroup $\bar{H}$ isomorphic to $\mathrm{A}\Gamma\mathrm{L}(1, p^h)$ then $\bar{H} = \bar{G}$. This corollary will be an essential tool in the proofs.

**Proposition 2.** *The autotopism collineation group $\mathcal{A}$ of $\pi$ contains a cyclic normal subgroup $N$ of order $d-1$ consisting of all $((0),[0])$-homologies of $\pi$, and the group $\mathcal{A}/N$ acts on the line $[0]$ as the semilinear group $\Gamma\mathrm{L}(1,q)$ in its usual transitive permutation representation on the affine line $\mathrm{AG}(1,q)$ over $\mathrm{GF}(q)$.*

**Proof.** The first statement is a consequence of (II). To prove the second part, we show that the only non-trivial collineation with Eq. (1) which has a fixed point on [0] distinct from $(0,0)$ is $x' = -x,\ y' = y$. Suppose that the mapping $y' = u^{d+1} y$ has such a fixed point. Then $u^{d+1} = 1$. Since g.c.d.$[d+1, q-1] = 2$, this yields $u^2 = 1$ and hence $u = -1$.

From this we infer that the group $U$ of all collineations with Eq. (1) acts on the line [0] as the group of permutations $y \to ay$, with $a$ ranging over all square elements of $\mathrm{GF}(q)^*$. Take a non-square element $b$ in $\mathrm{GF}(d)$. Since $\mathrm{GF}(q)/\mathrm{GF}(d)$ is an algebraic extension of odd degree, $b$ is still a non-square element of $\mathrm{GF}(q)$. Since the corresponding collineation $\sigma_b$ with Eq. (2) acts on [0] as the permutation $y \to by$, we see that $U$ together with $\sigma_b$ generate a group which acts on [0] as $\mathrm{GL}(1,q)$ on the affine line $\mathrm{AG}(1,q)$ over $\mathrm{GF}(q)$.

Taking account of the collineations with Eq. (4), we obtain that the group $W$ generated by all the collineations listed in (I) of Proposition 1 acts on [0] as the semilinear group $\Gamma\mathrm{L}(1,q)$ of $\mathrm{AG}(1,q)$. From this we infer that the translations fixing [0] together with $W$ generate a group which acts on [0] as the affine semilinear group $\mathrm{A}\Gamma\mathrm{L}(1,q)$ over $\mathrm{AG}(1,q)$. By Remark 2, the full collineation group fixing [0] is $WN$ where $N$ is the pointwise stabilizer of [0]. Since each collineation fixing [0] pointwise is a ((0), [0])-homology, Proposition 2 follows. □

**Remark 3.** It should be noticed that Proposition 2 can also be obtained as a special case from a general result due to Ganley and Jha [4] depending on the classification of finite simple groups.

## 2. The collineation group of the oval consisting of all absolute points of the natural orthogonal polarity of $\pi$

It is well known that the mapping which turns the point $P(u,v)$ into the line with equation $y=u*x-v$ is an orthogonal polarity of $\pi$ whose absolute points form the oval $\Omega=\{(t,\delta*(t*t))\colon t\in Q(+,*)\}\cup\{(\infty)\}$, where the constant $\delta$ is defined as the inverse of 2 in the multiplicative loop of $Q(+,*)$. An easy computation shows that $\Omega=\{(x,x^{d+1})\colon x\in \mathrm{GF}(q)\}\cup\{(\infty)\}$ in the model introduced in Section 1.

We state two properties of the collineation group of $\Omega$.

A point not on $\Omega$ is an internal or external point of $\Omega$ according as it lies on 0 or 2 tangents of $\Omega$. The internal points of $\Omega$ are affine points of $\pi$ and form a set $\mathscr{I}(\Omega)$ of size $\frac{1}{2}(q^2-q)$. Also, the set $\mathscr{E}(\Omega)$ of affine points of $\pi$ which are external points of $\Omega$ has size $\frac{1}{2}(q^2-q)$. Clearly, any collineation of $\pi$ fixing $\Omega$ leaves both $\mathscr{I}(\Omega)$ and $\mathscr{E}(\Omega)$ invariant.

**Proposition 3.** *The collineation group of $\pi$ fixing the oval $\Omega$ is isomorphic to the semi-linear group $\mathrm{A}\Gamma\mathrm{L}(1,q)$ and acts on the affine points of $\Omega$ as $\mathrm{A}\Gamma\mathrm{L}(1,q)$ in its natural 2-transitive permutation representation on the affine line $\mathrm{AG}(1,q)$ over $\mathrm{GF}(q)$.*

*Let $\mathscr{I}(\Omega)$ and $\mathscr{E}(\Omega)$ denote the set of all internal and affine external points of $\Omega$, respectively. Then the collineation group of $\pi$ fixing the oval $\Omega$ acts transitively on both $\mathscr{I}(\Omega)$ and $\mathscr{E}(\Omega)$.*

**Proof.** For $a\in\mathrm{GF}(q)$, the mapping $x'=x+a$, $y'=y+a^dx+ax^d+a^{d+1}$ is a collineation of $\pi$ fixing $\Omega$. These collineations form a group $\Sigma$ that acts on the affine points of $\Omega$ as the group of all translations $x\to x+a$ of $\mathrm{AG}(1,q)$. Also, each of the collineations with Eq. (1) or (4) fixes $\Omega$. These collineations form a group acting on the affine points of $\Omega$ as $\Gamma\mathrm{L}(1,q)$ on $\mathrm{AG}(1,q)$. Since no non-trivial collineation fixes $\Omega$ pointwise, from Remark 2 we obtain the first assertion.

The transitivity on $\mathscr{E}(\Omega)$ is a consequence of the 2-transitivity on the affine points of $\Omega$. To show the transitivity on $\mathscr{I}(\Omega)$, we also need a property of the group $U$ which we have pointed out in the proof of Proposition 2. According to this property, the group $U$ has only two non-trivial point-orbits on the line [0]. Since $U$ fixes both $\Omega$ and its point $(0,0)$, the first point-orbit consists of internal points (and the other one of external points). Hence $U$ together with the group $\Sigma$ introduced in the previous proof generate a collineation group which fixes $\Omega$ and acts transitively on $\mathscr{I}(\Omega)$. $\square$

## 3. The collineation group of the unital consisting of all absolute points of the natural unitary polarity of $\pi$

In this section we assume that the order $q$ of $\pi$ is a square.

Let $\theta : x \to x^{\sqrt{q}}$ be the involutory automorphism of GF($q$). According to [3, Theorem 8], the mapping which turns the point $P(u,v)$ into the line with equation $y = u^\theta * x - v^\theta$ is then a unitary polarity whose absolute points form the unital $\mathscr{U} = \{(u,v)\colon v + v^\theta - u^\theta * u = 0;\ u,v \in Q(+,*)\} \cup \{(\infty)\}$.

In the model introduced in Section 1, this unital becomes $\mathscr{U} = \{(x,y)\colon y + y^\theta - (x^{d+\theta} + x^{d\theta+1}) = 0;\ x,y \in \mathrm{GF}(q)\} \cup \{(\infty)\}$. To determine the collineation group of $\mathscr{U}$ we need to consider some new collineations as well as special classes of collineations defined in Proposition 1.

**Proposition 4.** (I) *The following mappings*

$$x' = ux, \qquad y' = u^{d+1}y, \quad u \in \mathrm{GF}(\sqrt{q})^* \tag{9}$$

$$x' = cx, \qquad y' = c^{\theta+1}y, \quad c \in \mathrm{GF}(d)^* \tag{10}$$

*are collineations fixing* $\mathscr{U}$.

(II) *The* $((0),[0])$*-homology group of* $\mathscr{U}$ *has order* $\sqrt{d}+1$, *and consists of the mappings*

$$x' = cx, \quad y' = y \quad \text{with } c \in \mathrm{GF}(d) \quad \text{and} \quad c^{\theta+1} = 1. \tag{11}$$

**Proof.** We only prove (II). By (II) in Proposition 1 each $((0),[0])$-homology has Eq. (11), and we must only note that $c^{\theta+1} = 1$ is the condition that $\mathscr{U}$ is fixed by such a homology. Since g.c.d.$[d-1, \sqrt{q}+1] = \sqrt{d}+1$, the number of mappings with Eq. (11) is $\sqrt{d}+1$. Thus (II) is proved. $\square$

**Proposition 5.** *Let* $\Phi$ *be the collineation group of* $\pi$ *fixing the unital* $\mathscr{U}$. *Then*

(i) $\Phi$ *has order* $mq\sqrt{q}(\sqrt{q}-1)(\sqrt{d}+1)$, *where* $q = p^m$, $p$ *prime*;

(ii) $\Phi$ *has a normal subgroup* $M$ *of order* $q\sqrt{q}$ *that acts on the affine points of* $\mathscr{U}$ *as a sharply transitive permutation group*;

(iii) *the stabilizer of an affine point of* $\mathscr{U}$ *under* $\Phi$ *has a normal cyclic subgroup of order* $(\sqrt{q}-1)(\sqrt{d}+1)$ *with cyclic complement of order* $m$.

**Proof.** We define $\Phi$ as the subgroup of the collineation group of $\pi$ which fixes $\mathscr{U}$. It is straightforward to check that $\Phi$ consists of all mappings

$$\varphi_{a,b} : (x, y) \to (x + a, y + a^{d\theta}x + a^{\theta}x^{d} + b) \quad \text{with } a^{d+\theta} + a^{d\theta+1} - b^{\theta} - b = 0.$$

Note that the latter condition is equivalent to $P(a,b) \in \mathscr{U}$. Thus, $M$ has order $q\sqrt{q}$. From this (ii) follows since no non-trivial element in $M$ fixes an affine point on $\mathscr{U}$.

Let $\Phi_0$ denote the stabilizer of the point $(0,0)$ under $\Phi$. Clearly the collineations in Proposition 4 and those with Eq. (4) belong to $\Phi_0$. To show that they generate $\Phi_0$, we argue as in Proposition 2, and consider their actions on the set $L$ of the common affine points of the line [0] and the unital $\mathscr{U}$. This set $L$ consists of all points $(0, y)$ such that $y^{\theta} + y = 0$ and $y \in \mathrm{GF}(q)$. The mappings with Eq. (9) act on $L$ as the permutations $y \to ay$ with $a$ ranging on the square elements of $\mathrm{GF}(\sqrt{q})^{*}$. To get a permutation $y \to cy$ with a non-square element $c \in \mathrm{GF}(\sqrt{q})$, choose $c \in \mathrm{GF}(d)$ such that $c^{\theta+1}$ is a non-square element in $\mathrm{GF}(\sqrt{d})$. Note that such an element must exist since $\theta$ induces a non-trivial involutory automorphism over $\mathrm{GF}(q)$ by virtue of the fact that $m/s = r$ with $r$ odd, see Section 1. This also yields that each non-square element of $\mathrm{GF}(\sqrt{d})$ is still non-square in $\mathrm{GF}(\sqrt{q})$. Hence the corresponding mapping with Eq. (10) acts on $L$ as desired. Thus the group generated by the collineations given in Proposition 4 together with those of Eq. (4) acts on $L$ as $\Gamma\mathrm{L}(1, \sqrt{q})$ on $\mathrm{GF}(\sqrt{q})$. By Remark 2, this is the whole group $\Phi_0$.

To finish the proof we note that the subgroup of $\Phi_0$ generated by the collineations (9), (10) and(11), consisting of the mappings

$$x' = ax, \qquad y' = a^{d+\theta}y, \quad a^{d-1} \in \mathrm{GF}(\sqrt{q})^{*},$$

is s cyclic group of order $(\sqrt{q} - 1)(\sqrt{d} + 1) = \text{g.c.d.}[(d - 1)(\sqrt{q} - 1), q - 1)]$. $\square$

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